ERRATIC EXTREMISTS

INDUCE

DYNAMIC CONSENSUS

(a new model for opinion dynamics)

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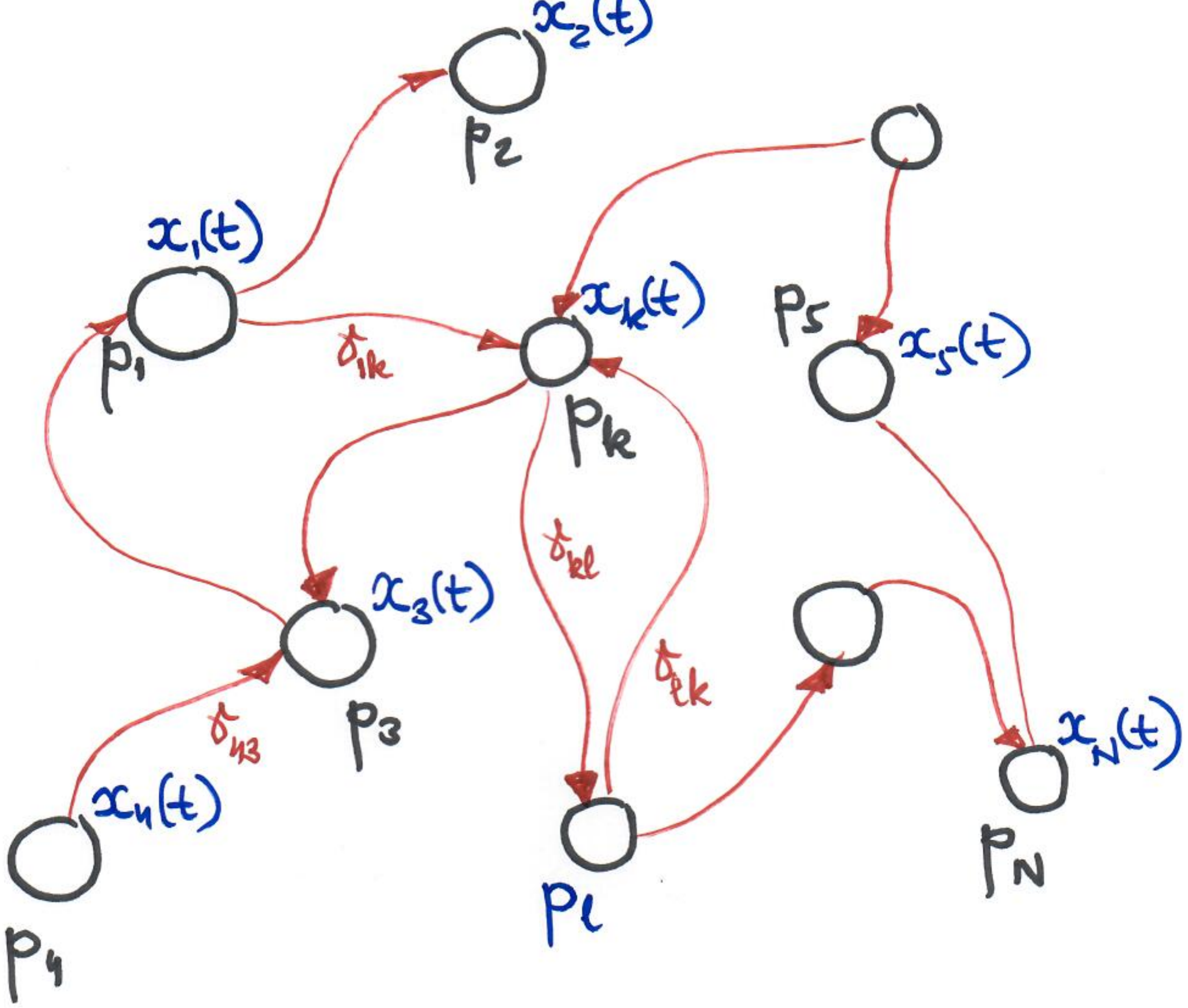
OPINION DYNAMICS

We consider a population of N agents and quantify their opinion/ideological position on some 'issue' by a real value. Then we postulate how each agent changes its opinion due to the opinions of the other agents.

Lots of opinion dynamics models are possible, and many were analysed over the years trying to explain various "emergent" phenomena in society.

MODELS:



STATE VECTOR $\mathbf{X}(t) = \begin{array}{c} p_1 \quad \cdots\cdots \quad p_N \\ [x_1(t) \; \cdots\cdots \; x_N(t)]^T \end{array}$

INITIAL STATE $\mathbf{X}(t=0)$

EVOLUTION PROCESS

$$\mathbf{X}(t+1) = \Psi\{\mathbf{X}(t) \,\vdots\, \text{parameters}\}$$

CLASSIC MODELS :

- (De Groot, 1974)

$$X(t+1) = \Gamma_{(\gamma_{ij}'s)} \cdot X(t)$$

fixed influence matrix

$X(t)$ - Linear Evolution controlled by the EigenStructure of Γ

- (Hegselmann-Krause, 2002)

$$x_k(t+1) = \frac{1}{|\mathcal{N}_k|} \sum_{l \in \mathcal{N}_k} x_l(t)$$

where $\mathcal{N}_k \triangleq \{ l \mid \|x_k(t) - x_l(t)\| < \varepsilon_k \}$

defines an ε_k-neighbourhood in opinions around the opinion of agent p_k.

(Usually $\varepsilon_k \equiv \varepsilon$ (GLOBAL))

Note that this is equivalent to making the matrix Γ dependent on $X(t)$ hence the evolution becomes nonlinear.

LOTS OF VARIATIONS ON THE CLASSICS

& LOTS OF OTHER APPLICATION AREAS

- OPINION DYNAMICS
- SWARM ROBOTICS
- DISTRIBUTED COMPUTING
- CROWD | FLOCK SIMULATIONS
- BIOLOGY OF SOCIAL ANIMALS

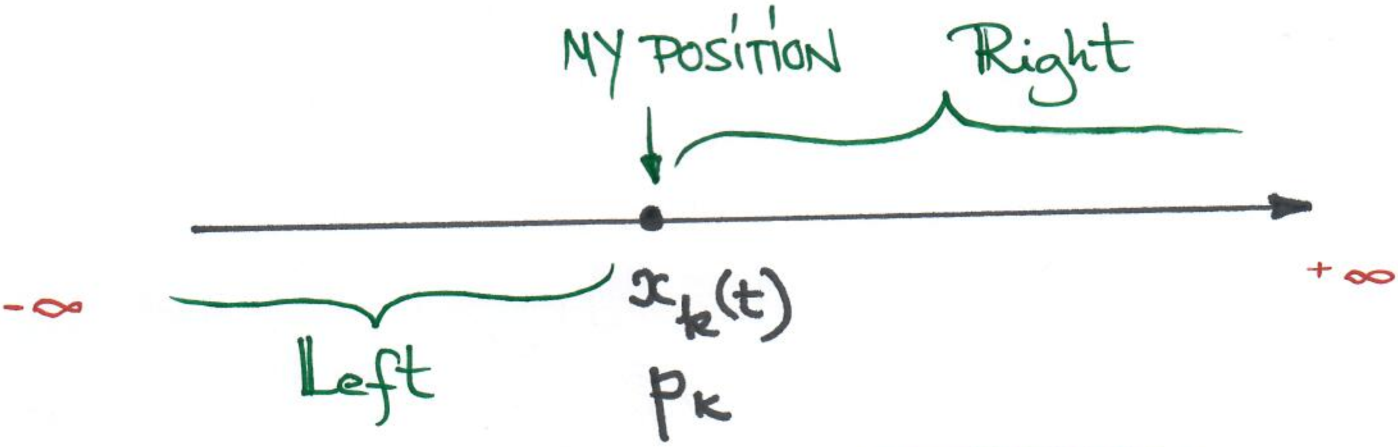
Topics: multi-agent consensus
gathering, rendez-vous
flocking, swarming
gossiping, geometric averageing
ant-colony modeling, swarms of locusts etc... etc... etc.

"Agents $p_1, p_2, \dots p_N$ in motion; they move due to neighbour's influence and do something together"

OUR MODEL

The agents $p_1, p_2, \dots p_N$ have initial opinions $[x_1(0)\ x_2(0)\ \dots\ x_N(0)]^T = \underline{X}(0)$

Agents are identical and indistinguishable and perform the following algorithm



- If $\exists$ others <u>both</u> to my Right and to my Left

  then
  $$x_k(t+1) \equiv x_k(t)$$

- If no others to my Right

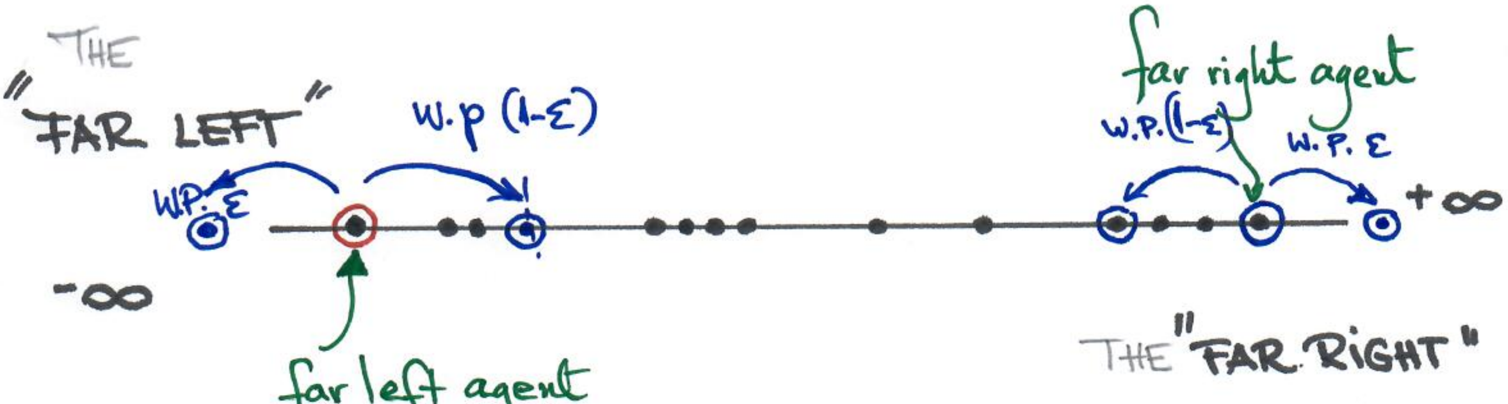
  then
  $$x_k(t+1) = \begin{cases} x_k(t)+1 & \text{W.P. } \varepsilon \\ x_k(t)-1 & \text{W.P. } (1-\varepsilon) \end{cases}$$

- If no others to my Left

  then
  $$x_k(t+1) = \begin{cases} x_k(t)+1 & \text{W.P. } (1-\varepsilon) \\ x_k(t)-1 & \text{W.P. } \varepsilon \end{cases}$$

The EVOLUTION of the state, i.e. of the ordered set of opinions is due to the motion of the two extremists.

The "Political" Map



The "erratic extremist" agents move, with probability $1-\varepsilon$ (high for small ε), towards the crowd but may, with probability ε (low), become more radical in their opinion.

Question:

What Happens in the Long Run?

SOME DEFINITIONS:

- We shall reorder the names of the agents after each step to have them at increasing locations on the line $\mathbb{R}$

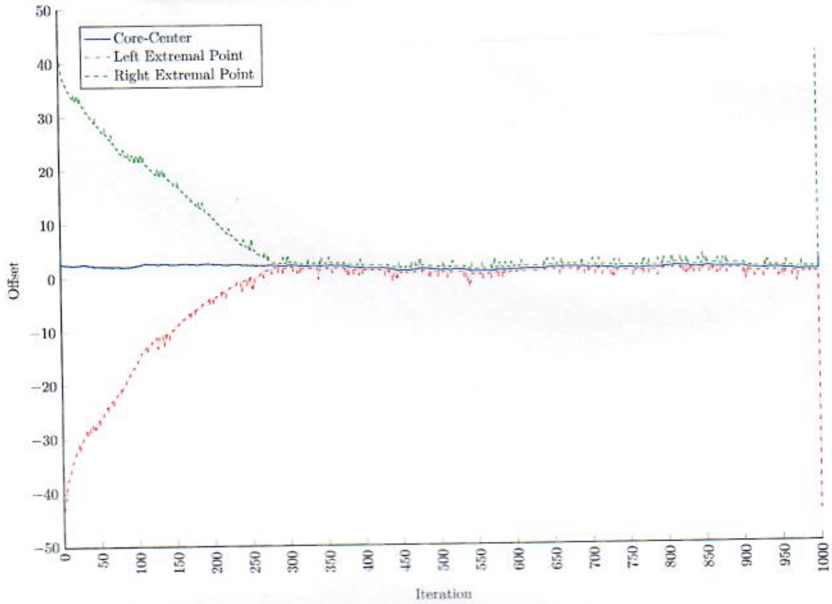
So we'll have at each time instant

with
- x_1 the LEFT ERRATIC EXTREMIST
- x_N the RIGHT ERRATIC EXTREMIST

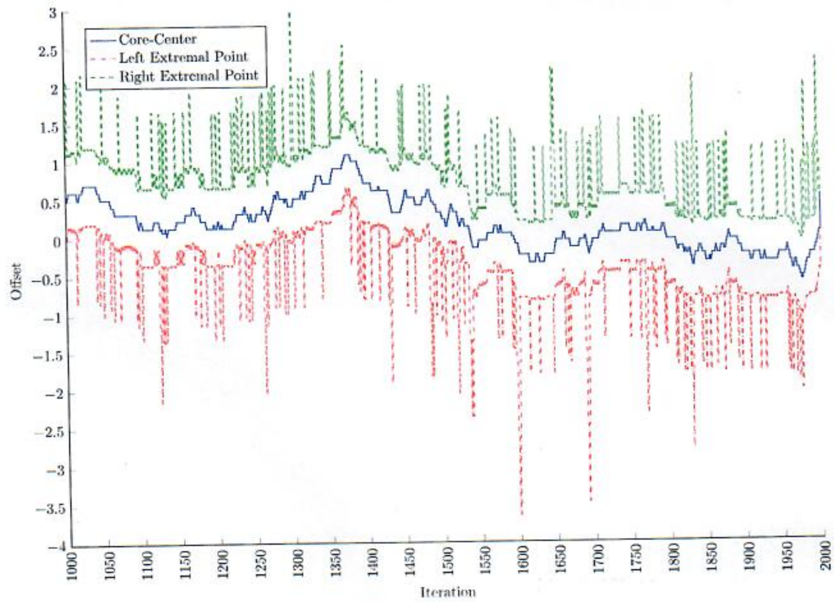
and
- the "core" group of moderates will span the interval $[x_2(t), x_{N-1}(t)]$

For simplicity (and w.l.o.g!) we'll assume that the initial locations of the agents have distinct fractional parts, which due to unit jumps will remain the same!

A TYPICAL SIMULATION RESULT IS THE FOLLOWING



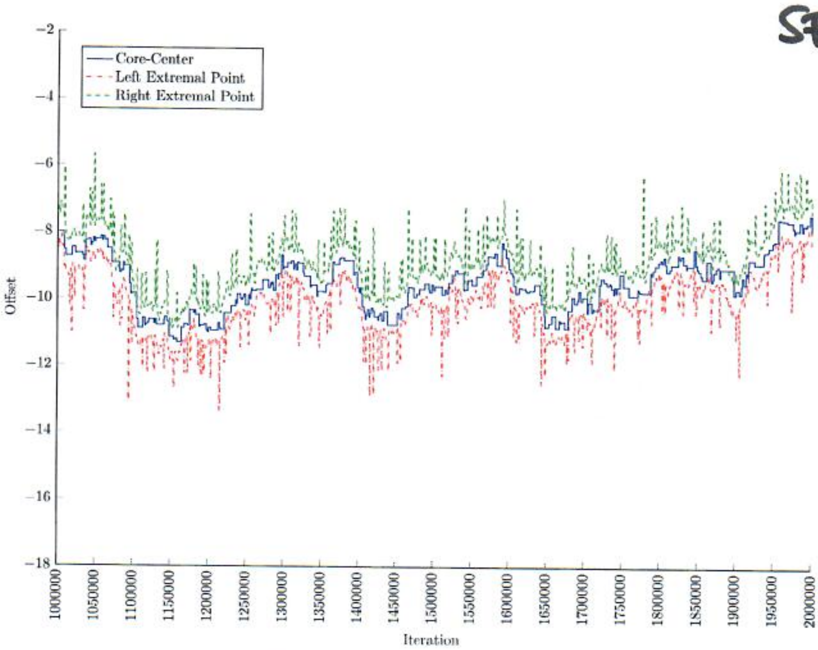
(a) First 1,000 iterations : The gathering process



(b) Iterations after gathering (T here was 1000 and the gathering happened at $t = 300$)

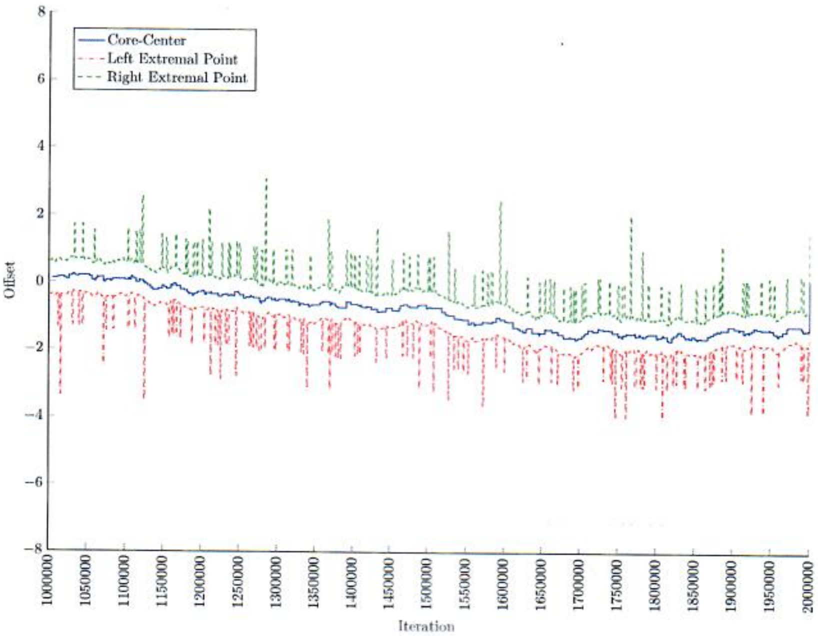
AS EXPECTED THE CORE GROUP GATHERS i.e. REACHES A (UNIT SPAN) CONSENSUS, AND THE CONSENSUS OPINION PERFORMS A RANDOM WALK!

VERY LONG TERM EVOLUTION OF CORE & EXTREMIST TO EXTREMIST SPAN



N=200

(a) Evolution of "core" and extremal agents (zoomed out). $N = 200$

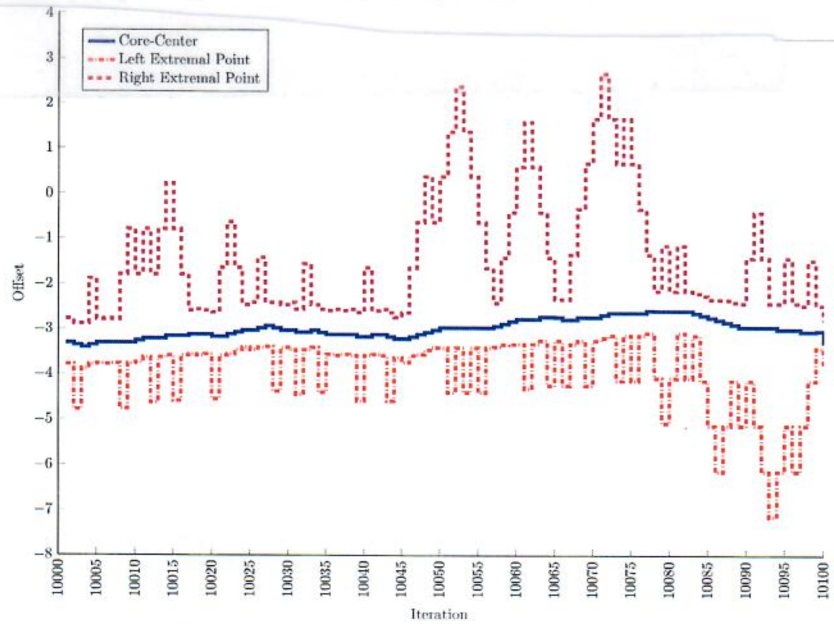


N=1000

(b) Evolution of "core" and extremal agents (zoomed out). $N = 1000$

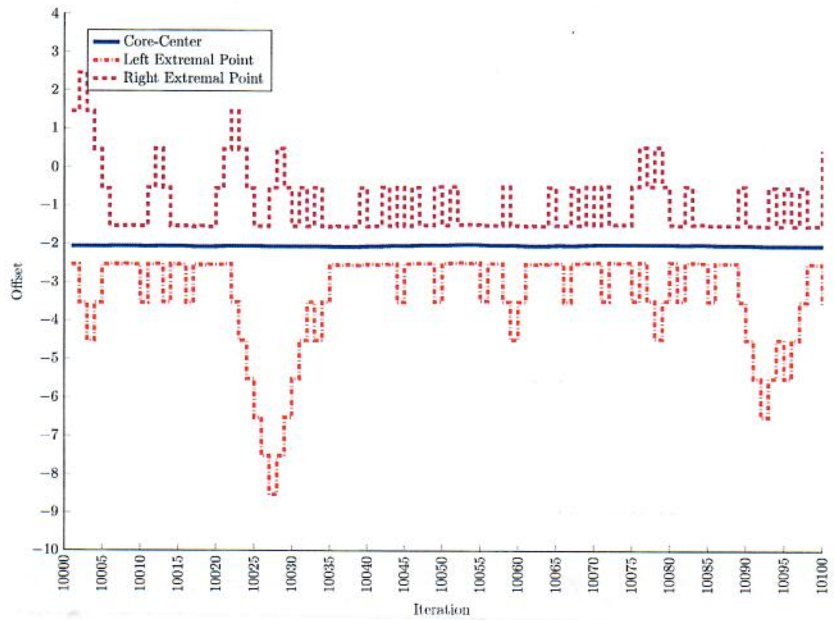
Figure [redacted]: Typical evolution of "core"'s center of mass location and extremal agents' locations in time (after gathering, starting from $T = 1,000,000$. Simulations done for $\varepsilon = 0.1$). Note that in both cases the initial center of mass of all agents was at 0. Note that the "inertia" of society is much higher when $N = 1000$ than in case $N = 200$.

... A ZOOM-IN ON THE EVOLUTION OF OPINIONS (AFTER 10000 ITERATIONS) FOR N = 21 AND N = 121 :



N=21

(a) "Core" and extremal agents (zoomed). $N = 21$



N=121

(b) "Core" and extremal agents (zoomed). $N = 121$

THE CORE OPINION IS FLUCTUATING MUCH LESS IN LARGE CROWDS!

The simulations show that:

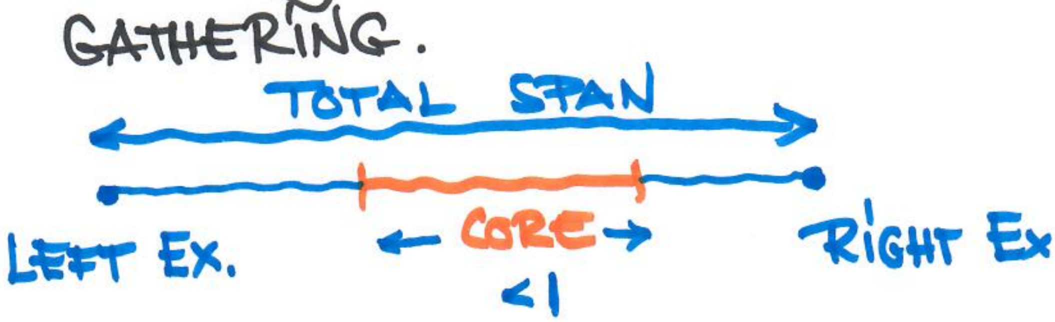
- The "core" of moderate opinions gathers to a span of less than 1
- The action of the "extremists" causes the "center of mass" of the core to fluctuate - performing a RANDOM WALK on the range of opinions $\mathbb{R}$
- The fluctuations of the consensus opinion are smoother in large populations, i.e. big crowds have big INERTIA and are more reluctant to change opinions, due to the extremists' actions!

What is of interest to us are the following issues:

- HOW LONG IT TAKES TO REACH CONSENSUS (i.e. for the core to gather!)
- HOW TO ASSESS THE ACTION OF THE EXTREMISTS after gathering: how do these move the crowd's average opinion
- HOW TO ANALYSE EXTENSIONS and GENERALIZATIONS TO "HIGHER DIMENSIONAL" PROBLEMS i.e. "VECTOR OPINIONS"

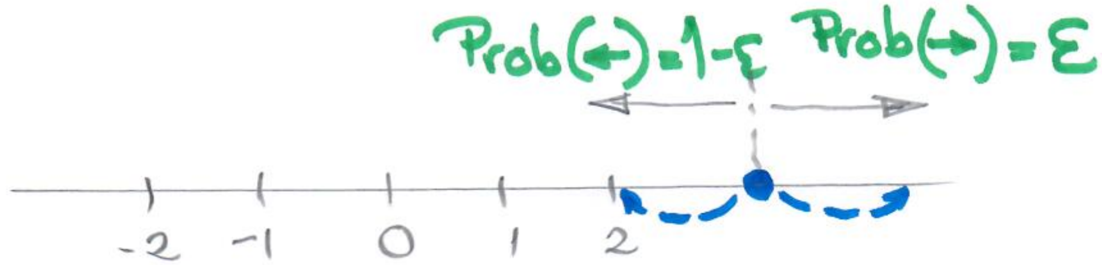
THE RESULTS SO FAR:

- BASED ON ANALYSIS OF BIASED RANDOM WALKS
- WE HAVE ESTIMATES for the EXPECTED TIME TO CONSENSUS AS FUNCTION OF
  - value of ε
  - initial span of agents
  - number of agents
- WE HAVE ESTIMATES FOR The Total Span of the Population (including the extremists) after GATHERING.



BASIC FACTS ON THE BIASED RANDOM WALK

$0 < \varepsilon < \frac{1}{2}$

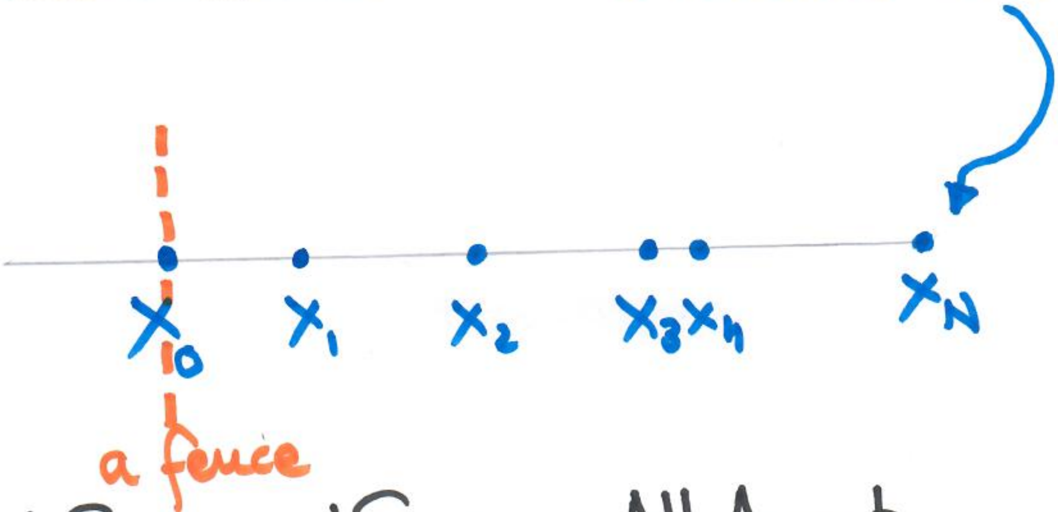


- $\text{Prob}(\text{walk hits } -1 \mid \text{starts at } 0) = 1$
- $\text{Prob}(\text{walk hits } +1 \mid \text{starts at } 0) = \frac{\varepsilon}{1-\varepsilon} < 1$
- $E(\#\text{steps to first hit } -1 \mid \text{starts at } 0) = \frac{1}{1-2\varepsilon}$
- $E(\#\text{steps to first hit } +1 \mid \text{starts at } 0) = \infty$
- $E\left(\text{farthest right excursion from } 0\right) \leq \frac{\varepsilon}{1-2\varepsilon}$

Proof: quite easy ~ Catalan Numbers based Generating Functions.

RESULTS

① • UNILATERAL ACTION BY RIGHT EXTREMIST



Right Extremist Sweeps All Agents across the "fence at x_0" in expected # of steps given by:

$$E(\#\text{steps for } \underbrace{x_0(0)}_{\text{fence}}\, x_1(0) \cdots x_N(0)) =$$

$$= \frac{1}{1-2\varepsilon} \sum_{k=1}^{N} \left(\lfloor x_k(0) - x_0(0) \rfloor + 1 \right)$$

and all agents "gather" to the interval

$(x_0(0)-1, x_0(0)]$ with probability 1

(2) • BILATERAL ACTION by BOTH "erratic" EXTREMISTS

(2.1) • If the "core" agents $p_2\, p_3 \cdots p_{N-1}$ are located at $x_2 < x_3 \ldots < x_{N-1}$ and $(x_{N-1} - x_2) \leq 1$ at time t, then we shall have that the new core at $t+1$

$$\left(x_{N-1}(t+1) - x_2(t+1)\right) \leq 1 .$$

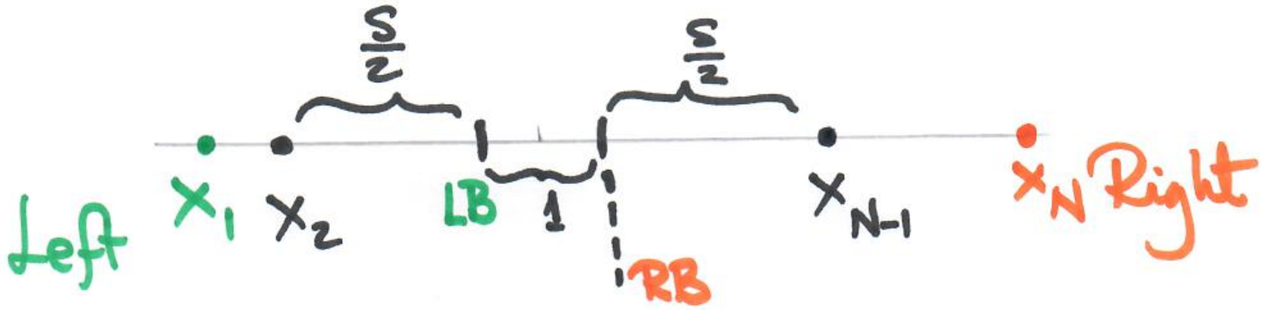
in other words the extremists' actions never expand the gathered core beyond 1

(2.2) • Analysis of the GATHERING to CONSENSUS is based on a decoupling idea, to overcome the difficulty in considering the joint action of the TWO EXTREMISTS

The De-COUPLING Idea:

• Suppose the "core span" is bigger than 1. Then we can write

$$x_{N-1} - x_2 = 1 + S$$



Locate two "beacons" as shown LB and RB !
These are some places that enable us to decouple the activities of the far Right and far Left extremists.

• the RIGHT extremists' actions will not be felt in the left while ∃ agents to the right of RB beacon and similarly
• the LEFT will not be felt in the right while ∃ agents to the left of the LB beacon.

When one of the extremists first "clears" its area between the initial core agents located in the $S/2$ - size region defined by its beacon (and we know this will happen w.p. 1 in finite expected time from the UNILATERAL ACTION Result; and indeed his action will never feel the other side, because he is first to clear its region!) the CORE shrank in span by $S/2$! The expected number of time steps is bounded by

$$E(T) < \frac{1}{1-2\varepsilon}\left(\underbrace{(N-2)}_{\text{worst case analysis}}\underbrace{\left\lceil\frac{S}{2}\right\rceil}_{\text{most possible steps}} + \underbrace{(x_N - x_1 - 1)}_{\text{TOTAL SPAN at START}}\right)$$

Repeating the argument while the core is bigger than 1 in span, and realizing that each step reduces the core by a "quantum" step necessarily bigger (strictly) than zero (recall that we have a finite # of agents, hence a finite set of differences in their fractional values!), we get that the core will shrink from the initial $1 + S(0)$ to $1 + \frac{S(0)}{2}$ to $1 + \frac{S(0)}{4}$ ... etc untill $\boxed{\frac{S(0)}{2^k}}$ will be smaller than the quantum (hence a finite # of iterations).

We obtain the following result:

Theorem

Given $x_1(0) \dots x_N(0)$ the initial ordered set of opinions, define

$$x_{N-1}(0) - x_2(0) \triangleq 1 + S_0$$

and $$x_N(0) - x_1(0) \triangleq M_0$$

Then the erratic extremists will induce consensus, w.p.1, in finite expected time upper bounded by

$$\boxed{E(T_{consensus}) < \frac{1}{1-2\varepsilon}\left(N\cdot\left(S_0 + \left\lceil \log_2 \frac{S_0}{d}\right\rceil\right) + M_0 - S_0 - 1\right)}$$

(where d is the smallest interval defined by the fractional parts of $x_1(0) \dots x_N(0)$!).

... and we also have that the span of the opinions, i.e. $x_N(t) - x_1(t)$ as time evolves, becomes the sum of the "core span" (which is less than 1) and two random excursions of biased random walks in the directions opposite to their bias. The result is

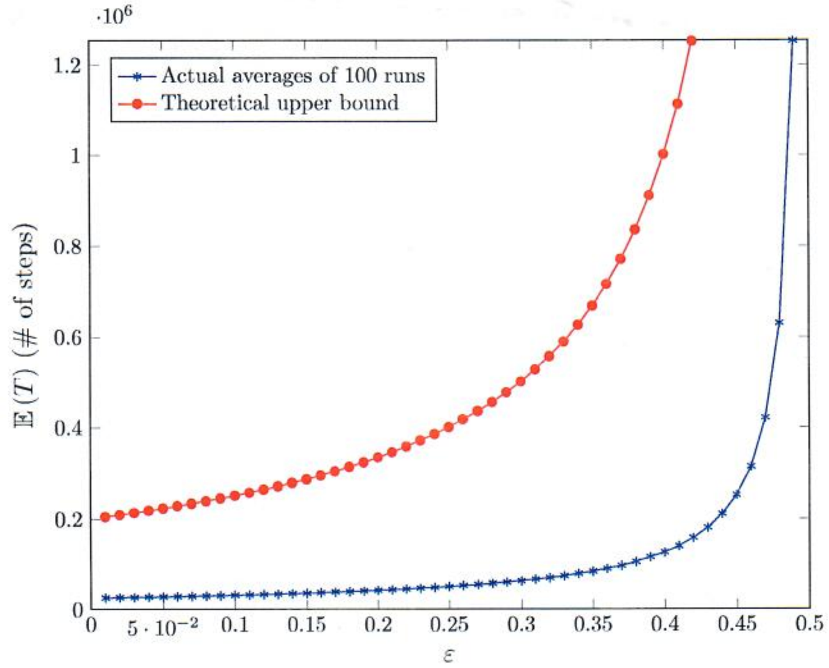
- $E\left(x_N(t) - x_1(t)\right) \leq 1 + \frac{2\varepsilon}{1-2\varepsilon}$

or using the Markov inequality

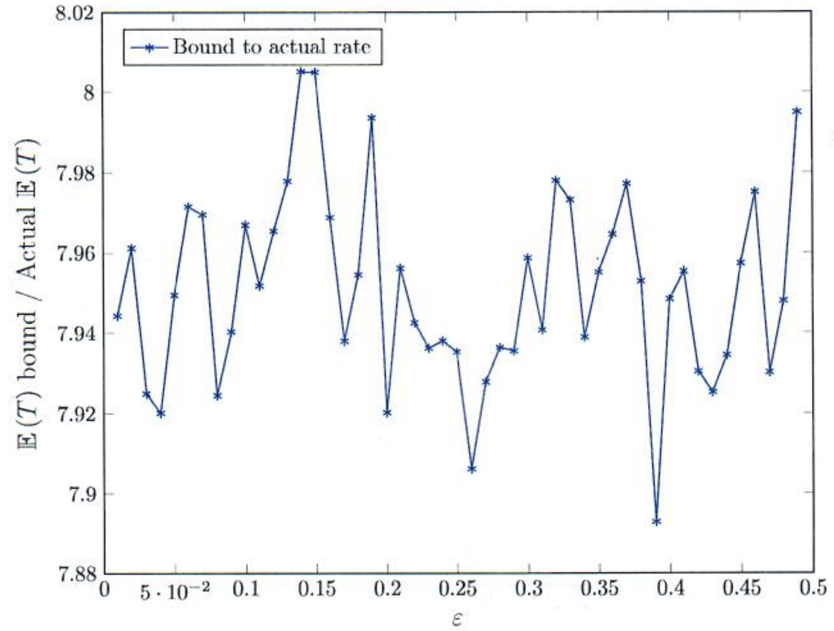
$$\Pr\{x_N(t) - x_1(t) \geq k\} \leq \frac{1}{k}$$

A more careful analysis provides:

$$\Pr\{\{x_N(t) - x_1(t)\} < k\} > 1 - k\left(\frac{\varepsilon}{1-\varepsilon}\right)^{k-2}$$



(a) $N = 400$, $S_0 = 500$



(b) $N = 400$, $S_0 = 500$

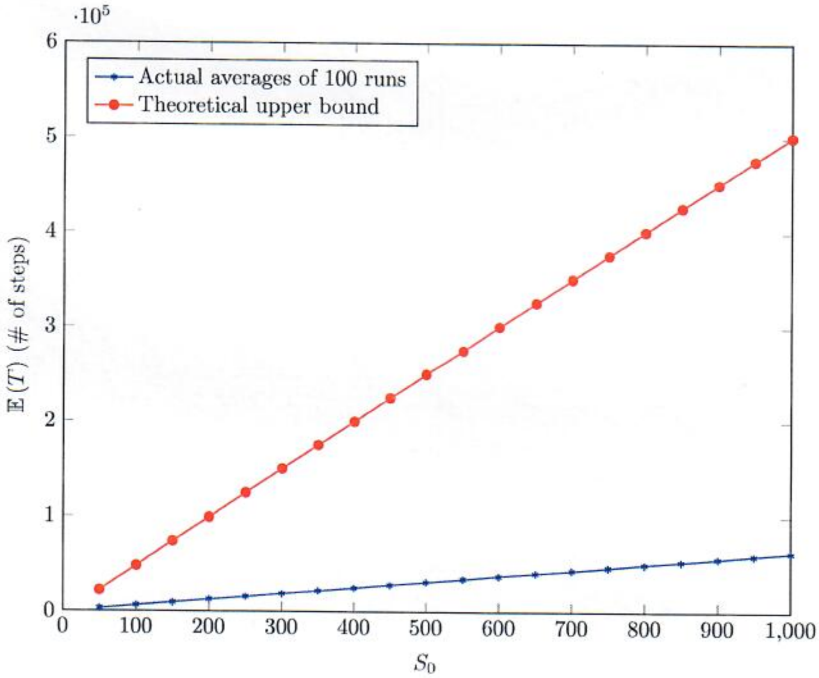
Bound/Empirical Result for 100 runs

$\cong$ 8 times (better)

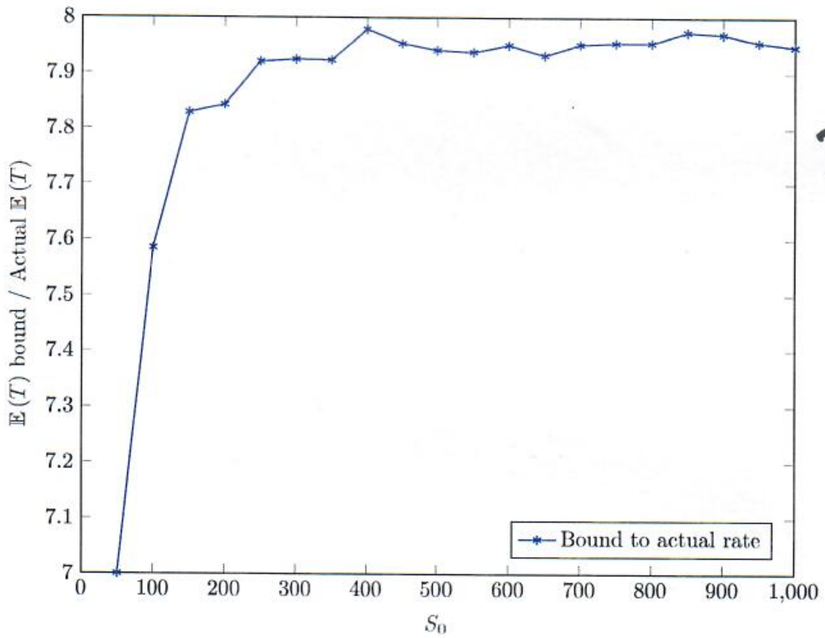
Figure [redacted] (a) Convergence times as a function of probability of motion in the wrong direction(ε), $N = 400$, $S_0 = 500$. (b) Theoretical upper bound to measured convergence time ratio vs ε. Each point on the actual results' line is an average of 100 different simulations with the same parameters set.

$E(T_{\text{CONSENSUS}})$ [redacted] vs (ε)

E(T CONSENSUS) vs S₀ (initial core span)



(a) $N = 400$, $\varepsilon = 0.1$

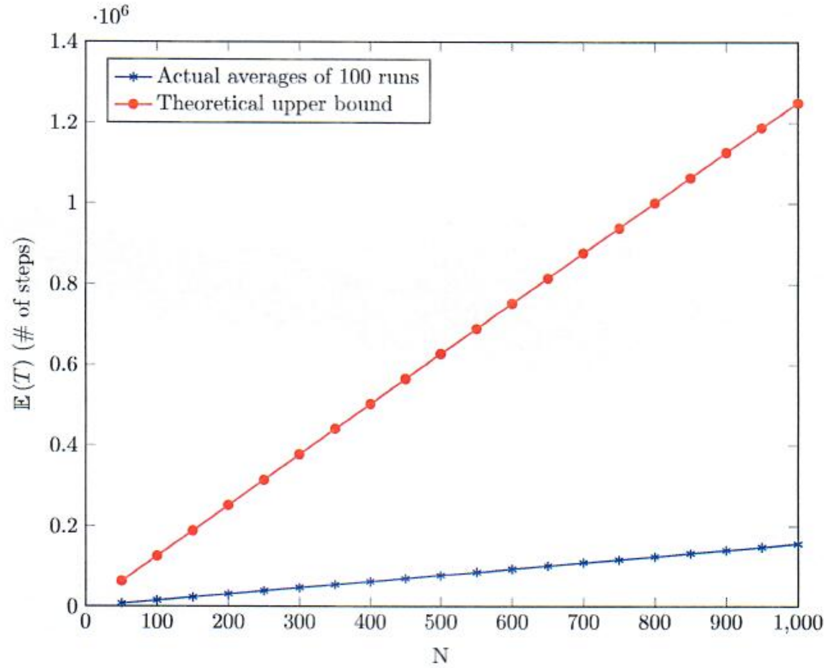


Bound/Empirical
≅ 8 times
(better)

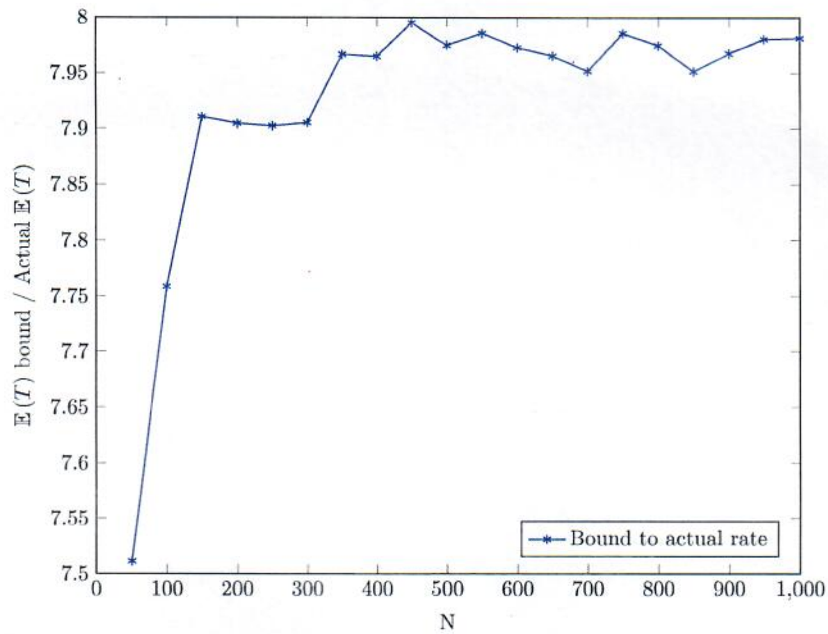
(b) $N = 400$, $\varepsilon = 0.1$

Figure [redacted] (a) Convergence times as a function of initial span (S_0). (b) Ratio between the theoretical upper bound to measured convergence time vs S_0. Each point on the actual results' line is an average of 100 different simulations with the same parameter set.

E(T CONSENSUS) vs N (number of agents)



(a) $S_0 = 1000$, $\varepsilon = 0.1$

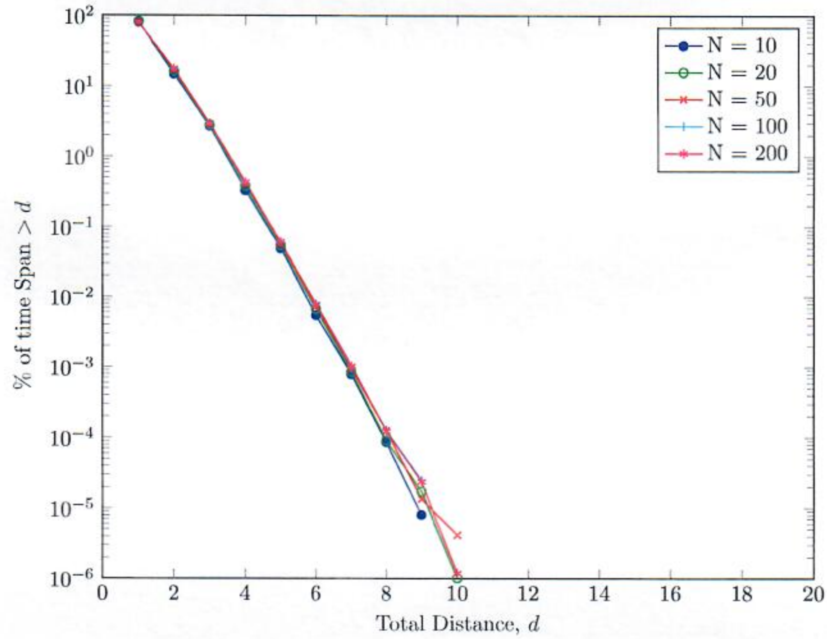


(b) $S_0 = 1000$, $\varepsilon = 0.1$

Bound/Empirical
≅ 8 times
(better)

Figure [redacted] (a) Convergence times as a function of number of agents (N). (b) Theoretical upper bound to measured convergence time vs N. Each point on the actual results' line is an average of 100 different simulations with the same parameters set.

Probability of TOTAL SPAN being more than d units.



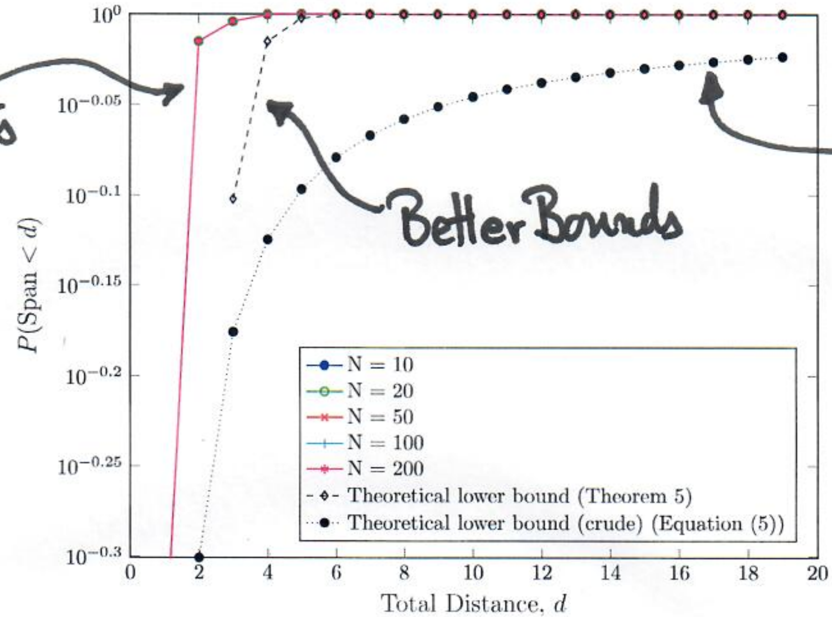
(a) Long-term distribution of agent span

Experiments

$\varepsilon = 0.1$

Better Bounds

Crude result



(b) Cumulative distribution of spans

Figure [redacted] (a) Long-term (steady-state) distribution of agents' total span, (b) Long-term cumulative distribution of total span with lower bounds from subsection 3.4 and subsection 4.3. The simulations were done for different values of N and $\varepsilon = 0.1$.

CONCLUDING REMARKS

- WE SHOWED THAT ERRATIC EXTREMISTS INDUCE CONSENSUS IN FINITE TIME AND WE SAW THAT THE CONSENSUS OPINION FLUCTUATES LIKE RANDOM WALK.
- The right wing extremist effectively pushes the gathered crowd of moderates toward the LEFT while
- The left wing extremist effectively pushes the gathered crowd towards the RIGHT.